RANDOM MAGNETIC IMPURITIES AND THE LANDAU PROBLEM

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Abstract : The 2-dimensional density of states of an electron is studied for a Poissonian random distribution of point vortices carrying α flux in unit of the quantum of flux. It is shown that, for any given density of impurities, there is a transition, when $\alpha \simeq 0.3 - 0.4$, from an "almost free" density of state -with only a depletion of states at the bottom of the spectrum characterized by a Lifschitz tail- to a Landau density of state with sharp Landau level oscillations. Several evidences and arguments for this transition -numerical and analytical- are presented.

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I. Introduction :

One considers a 2-dimensional model for an electron of electric charge e and of mass m subject to a random magnetic field. Here, random magnetic field means a Poissonian[3] random distribution $\{\vec{r}_i, i = 1, 2, ..., N\}$ of fixed infinitely thin vortices carrying a flux ϕ, modeling magnetic impurities, and characterized by the dimensionless Aharonov-Bohm $(A - B)$ coupling $\alpha = e\phi/2\pi = \phi/\phi_o$.

The question we ask is about the effect of disorder on the energy level density of an electron -test particle- averaged over the random position of the vortices [1]. In the thermodynamic limit $N \to \infty, V \to \infty$ for a distribution of vortices of density $\rho = N/V$, one might naively argue that the average magnetic field, $< B >= \alpha\rho\phi_o$, is expected to become meaningful in the limit $\rho \to \infty, \alpha \to 0$, with $\rho\alpha$ kept finite. On the other hand, if ρ is finite, and α non vanishing, corrections due to disorder should exhibit non trivial magnetic impurity signatures, like broadening of Landau levels and localization.

In a first paper [2] on this problem, two approaches were used:

- A path integral Brownian motion analysis where the problem at hand is mapped, after averaging on disorder, to a study of winding properties of Brownian curves.

- A quantum mechanical formulation, where the contact hard-core boundary conditions at the location of the impurities where properly taken into account by an appropriate wave function redefinition, allowing for an analytical averaging on the random singular $A - B$ interactions.

The main results were :

- in the presence of a constant external magnetic field, if the total (external + average) magnetic field is strong enough so that one can neglect the coupling between the low-

[3] One could consider as well other types of probability distribution - Gaussian for example. For the sake of simplicity, we concentrate only on the Poisson case.

est Landau level and the excited Landau levels by the random component of the vortex distribution, the system was best described when projected in the LLL. Since one has in view a sufficiently dilute gas of electrons compared to the available quantum states in the LLL -the fractional Quantum Hall regime-, such a restriction is licit. In this situation, the quantum problem was explicitly mapped on a problem of random δ impurities, where the average density of states happened to be known [3].

- in the average magnetic field limit $\alpha \to 0, \rho \to \infty$, a global zero point energy shift $\frac{|e<B>|}{2m} =< \omega_c >$ materialized in the Landau spectrum of the average magnetic field $< B >$. The origin of this shift was traced back to the hard-core boundary conditions at the location of the impurities [4].

- in the case $\alpha = 1/2$ and ρ finite, on the other hand, Brownian numerical simulations showed that the sole effect of the impurities was a depletion of states at the bottom of the spectrum, characterized by a Lifschitz tail in the average density of states, caused by the isolated impurities.

An additional interesting result was that the average density of states $< \rho(E, \alpha, \rho) >$ happened to be a function of E/ρ and α only (this is also true in presence of an external magnetic field). The $< \rho(E/\rho, \alpha) >$ scaling implies that the impurity density ρ is not a relevant parameter, and can be arbitrarily set to a given value, since changing its value amounts simply to a rescaling of the energy unit. A consequence is a more precise definition of the limit where the average magnetic field becomes meaningful. **It should happen, for any given finite ρ, when α becomes sufficiently small, independently of the value of ρ. Thus one expects a critical value α_c where a transition occurs from an almost free density of states with a Lifschitz tail at the bottom of the spectrum ($\alpha_c < \alpha < 1/2$), to a Landau like density of states, with Landau oscillations, i.e. Landau levels separated by a Landau gap ($0 < \alpha < \alpha_c$).**

A semiclassical understanding of this transition consists in taking an electron with a Fermi velocity v_F, subject to the average magnetic field $< B >= \rho\alpha\phi_o$. A typical cyclotron orbit radius is $R \simeq v_F/(\rho\alpha)$. For a given ρ, one has $R \simeq 1/\alpha$, thus the smaller is α, the larger is the number of magnetic impurities enclosed by the cyclotron orbit, and therefore the more accurate is description in terms of the average magnetic field. To summarize, for a given finite ρ, the smaller is α, the better is $< B >$, eventhough it is smaller and smaller. However, there is no other magnetic scale in the problem to compare it with.

An important consequence of the Lifschitz tail - Landau levels transition should be a non conducting-conducting transition, since one expects localization due to disorder when $\alpha > \alpha_c$, and extended states in the Landau regime when $\alpha < \alpha_c$.

The aim of the present paper is to show that the critical value at which the transition occurs is $\alpha_c \simeq 0.3 - 0.4$.

We will present two types of evidence for this result :

- Numerical evidences

i) for the specific heat, where a Brownian motion numerical study indicates a transition from an almost free type to a Landau type specific heat when $\alpha_o^{num} \simeq 0.28$.

ii) for the density of state, where, under some reasonnable simplifying assumptions, the transition is shown to occur at $\alpha_c \simeq 0.35$.

- Analytical evidences :

i) one can explicitly show that the specific heat transition is possible only when Landau oscillations have already begun, implying that $\alpha_c > \alpha_o$, but close to α_o.

ii) quantum mechanical evidence using the impurity cluster expansion of the average partition function for an arbitrary large number of impurities. At a given order ρ^N, for a test particle subject to a given number N of impurities described by the Hamiltonian H_N, the average partition function $< Tr \exp(-\beta H_N) - Tr \exp(-\beta H_0) >$ has been computed

as a power series in α. One recovers, at order $\rho^N \alpha^N$, the leading order in α, the partition function of the average magnetic field. Higher order corrections can be computed : for $N = 2$, the diagrammatic expansion has been done up to order α^4. We will show that this is all what is needed to obtain $\alpha_o \simeq 0.29$, which is indeed very close to the numerical result $\alpha_o^{num} = 0.28$.

II. Definition of the model :

The system is peridodic in α with period 1, so one can always take $\alpha \in [0, 1]$. In the absence of an external magnetic field, there is no privileged orientation to the plane, therefore the system is symetric wrt $\alpha = 1/2$, and one can restrict $\alpha \in [0, 1/2]$. It follows that any physical quantity of interest should depend on $\alpha(1 - \alpha)$ only.

II.a Path integral Brownian approach :

One starts [2] from a square lattice of $\mathcal{N}$ squares of size a^2, in which point magnetic impurities are randomly dropped. Let N_i be the number of vortices dropped on square i. A random configuration $\{N_i\}$ will be realized with the probability

$$P(\{N_i\}) = \frac{N!}{\mathcal{N}^N \prod_{i=1}^{\mathcal{N}} N_i!} \rightarrow_{\mathcal{N}\rightarrow\infty} \prod_i^{\mathcal{N}} \frac{(\rho a^2)^{N_i} e^{-\rho a^2}}{N_i!} \tag{1}$$

with $N/\mathcal{N} = \sum_{i=1}^{\mathcal{N}} N_i/\mathcal{N} = \rho a^2$. In the thermodynamic limit, this is nothing but a Poissonian distribution. In order to compute the average level density $< \rho(E) >$, one focuses, in the thermodynamic limit $\mathcal{N} \rightarrow \infty$, on the one-electron average partition function per unit volume (for an electron of unit mass and charge)

$$Z = Z_o < e^{i \sum_{i=1}^{\mathcal{N}} 2\pi n_i N_i \alpha} >_{\{C, N_i\}} \tag{2}$$

where $\{C\}$ is the set of L steps closed random walks, and n_i is the number of times the square i has been wound around by a given random walk in $\{C\}$, i.e. its winding number. $Z_o = \frac{1}{2\pi t}$ is the free partition function per unit volume, with t (the inverse temperature β)

the length of the curve ($2t = La^2, e = m = 1$). (2) is obviously invariant when α is shifted by an integer, and when $\alpha \to -\alpha$ (because n_i comes always with $-n_i$), so one can always restrict to $0 < \alpha < 1/2$. Averaging Z with (1) one gets

$$Z = Z_o < e^{\rho \sum_n S_n (e^{i2\pi\alpha n} - 1)} >_{\{C\}} \tag{3}$$

where S_n stands for the arithmetic area of the n-winding sector [5] of a given random walk in $\{C\}$. Eq. (3) is still true in the limit $a \to 0, L \to \infty$, with t fixed, i.e. for Brownian curves in the plane, yielding a path integral formulation for the problem at hand. Extracting the variable t, it rewrites as

$$Z = Z_o \int e^{-\rho t(S+iA)} P(S,A) dS dA \equiv Z_o < e^{-\rho t(S+iA)} >_{\{C\}} \tag{4}$$

where S and A are defined as

$$S = \frac{2}{t} \sum_n S_n \sin^2(\pi\alpha n); \quad < S >= \pi\alpha(1-\alpha) \tag{5}$$

$$A = \frac{1}{t} \sum_n S_n \sin(2\pi\alpha n); \quad < A >= 0 \tag{6}$$

$P(S,A)$ is the joint probability distribution of the random variables A and S. One has used the general property of Brownian curves that S_n scales like t (remind that $< S_n >= t/(2\pi n^2)$ [5]), so that the variables S and A are actually t independent. This implies that the average partition function Z in (4) has the form $F(\rho t)/t$, and thus its inverse Laplace transform, the average density of states, is necessarily a function of E/ρ and α.

II.b Microscopic Quantum Hamiltonian :

The Hamiltonian of an electron of mass m and charge e subject to the potential vector of N vortices at position $\vec{r}_i$ ($\vec{k}$ is the unit vector perpendicular to the plane) reads [2]

$$H_N = \frac{1}{2m} \left(\vec{p} - \sum_{i=1}^{N} \alpha \frac{\vec{k} \times (\vec{r} - \vec{r}_i)}{(\vec{r} - \vec{r}_i)^2} \right)^2 \tag{7}$$

Because of periodicity in α, one can always restrict α to $\alpha \in [-1/2, +1/2]$. We consider in the thermodynamic limit the Poisson probability distribution $dP(\vec{r}_i) = d\vec{r}_i/V$.

The system described by the Hamiltonian (7) is not yet entirely defined. Boundary conditions on the wave functions have to be specified when the electron comes close to an impurity. This can be achieved in a non ambiguous way by adding to the pure Aharonov-Bohm Hamiltonian the contact terms

$$H_N^{\pm} = \frac{1}{2m}\left(\vec{p} - \sum_{i=1}^{N} \alpha \frac{\vec{k} \times (\vec{r} - \vec{r}_i)}{(\vec{r} - \vec{r}_i)^2}\right)^2 \pm \sum_i \frac{\pi|\alpha|}{m} \delta^2(\vec{r} - \vec{r}_i) \tag{8}$$

These contact terms amount to couple the spin-up $(+)$ or spin-down $(-)$ degree of freedom [6] of the electron endowed with a magnetic moment $\mu = -\frac{e}{2m}\alpha/|\alpha|$ (thus an electron with gyromagnetic factor $g = 2$), to the infinite magnetic field inside the flux-tubes. Their origin can also be understood in a perturbative framework : the original Aharonov-Bohm spectrum with vanishing wavefunction at the location of the magnetic impurities (a particular self adjoint extension describing impenetrable vortices : hard-core) can be perturbatively obtained if and only if the contact terms with the $(+)$ sign are taken into account, whereas the contact terms with the opposite $(-)$ sign correspond to a different self-adjoint extension, where the the wavefunction is singular at the location of the vortices (the particle is attracted inside the vortices : attractive-core). Note that in the Brownian path integral formulation, only the hard-core case can be described, due to the fact that, by definition, a given Brownian path has no chance to pass through a given impurity location.

The contact terms happen to be crucial for the averaging on the disorder : consider the nonunitary wavefunction redefinition [7]

$$\psi_N^{\pm}(\vec{r}) = \prod_{i=1}^{N} |\vec{r} - \vec{r}_i|^{\pm|\alpha|} \tilde{\psi}_N^{\pm}(\vec{r}) \tag{9}$$

to obtain the Hamiltonian $\tilde{H}_N^{\pm}$ acting on $\tilde{\psi}_N^{\pm}(\vec{r})$ where the impurity potential now reads

$$\tilde{H}_N^{\pm} = -\frac{2}{m}\partial_{\bar{z}}\partial_z - \frac{\alpha \pm |\alpha|}{m}\sum_{i=1}^{N}\frac{\partial_z}{\bar{z}-\bar{z}_i} + \frac{\alpha \mp |\alpha|}{m}\sum_{i=1}^{N}\frac{\partial_{\bar{z}}}{z-z_i} \tag{10}$$

which rewrites if $\alpha > 0$ as

$$\tilde{H}_N^{+} = -\frac{2}{m}\partial_{\bar{z}}\partial_z - \frac{2\alpha}{m}\sum_{i=1}^{N}\frac{\partial_z}{\bar{z}-\bar{z}_i} \tag{11}$$

$$\tilde{H}_N^{-} = -\frac{2}{m}\partial_{\bar{z}}\partial_z + \frac{2\alpha}{m}\sum_{i=1}^{N}\frac{\partial_{\bar{z}}}{z-z_i} \tag{12}$$

and if $\alpha < 0$

$$\tilde{H}_N^{+} = -\frac{2}{m}\partial_{\bar{z}}\partial_z + \frac{2\alpha}{m}\sum_{i=1}^{N}\frac{\partial_{\bar{z}}}{z-z_i} \tag{13}$$

$$\tilde{H}_N^{-} = -\frac{2}{m}\partial_{\bar{z}}\partial_z - \frac{2\alpha}{m}\sum_{i=1}^{N}\frac{\partial_z}{\bar{z}-\bar{z}_i} \tag{14}$$

One sees in both $\tilde{H}_N^{\pm}$ cases that there is again no distinction to be made between $\alpha \in [0, 1/2]$ and $\alpha \in [-1/2, 0]$. Indeed, the impurity potential are hermitian conjugate the one from the other, but perturbative computations in α are always such that real results are obtained. So one can always restrict to $\alpha \in [0, 1/2]$. Both $\tilde{H}_N^{\pm}$ cases should be considered in principle. However, it is easy to see that they are equivalent, except for a global shift $+ < \omega_c >$ in the hard-core $\tilde{H}_N^{+}$ case, and $- < \omega_c >$ in the attractive-core $\tilde{H}_N^{-}$ case. Indeed, consider, instead of (9), the wavefunction redefinition where the average magnetic field Landau pre-exponential factor has been extracted [2]

$$\psi_N^{\pm}(\vec{r}) = e^{\mp\frac{1}{2}m<\omega_c>r^2}\prod_{i=1}^{N}|\vec{r}-\vec{r}_i|^{\pm\alpha}\tilde{\psi}_N^{\pm}(\vec{r}) \tag{15}$$

to get (in the thermodynamic limit $N \to \infty$)

$$\tilde{H}_N^{\pm} = \pm < \omega_c > + H_{\pm<B>} + V^{\pm}(\alpha) - < V^{\pm}(\alpha) > \tag{16}$$

where $H_{<B>}$ is the Landau Hamiltonian for the average $< B >$ field and the impurity potential reads

$$V^+(\alpha) - < V^+(\alpha) > = -\frac{2\alpha}{m}\sum_{i=1}^{N}\frac{\partial_z}{\bar{z}-\bar{z}_i} + 2 < \omega_c > z\partial_z + \sum_{i=1}^{N} < \omega_c > \alpha\frac{\bar{z}}{\bar{z}-\bar{z}_i} - m < \omega_c >^2 \bar{z}z \tag{17}$$

($V^-(\alpha) - < V^-(\alpha) >$ is obtained by taking the hermitian conjugate of $V^+(\alpha) - < V^+(\alpha) >$ and $\alpha \to -\alpha$). Clearly, in the limit with no disorder $\alpha \to 0$, one obtains the Landau Hamiltonian for the average $< B >$ field with a $\pm < \omega_c >$ shift.

The Hamiltonians $H_N^{\pm}$ and $\tilde{H}_N^{\pm}$ are equivalent, and can be indifferently used for computing the partition function or the density of states. However, interactions with two magnetic impurities have disappeared from $\tilde{H}_N^{\pm}$, greatly simplifying the average on the disorder, which can be easily done using the identities

$$\int d\bar{z}_i dz_i \frac{1}{\bar{z}-\bar{z}_i}\partial_z = \pi z\partial_z \tag{18}$$

$$\int d\bar{z}_i dz_i \frac{1}{\bar{z}-\bar{z}_i}\partial_z \frac{1}{\bar{z}'-\bar{z}_i}\partial_{z'} = \pi(\frac{z}{\bar{z}'-\bar{z}} + \frac{z'}{\bar{z}-\bar{z}'})\partial_z\partial_{z'} \tag{19}$$

$$\int d\bar{z}_i dz_i \frac{1}{\bar{z}-\bar{z}_i}\partial_z \frac{1}{\bar{z}'-\bar{z}_i}\partial_{z'} \frac{1}{\bar{z}"-\bar{z}_i}\partial_{z"} =$$

$$\pi(\frac{z}{(\bar{z}'-\bar{z})(\bar{z}"-\bar{z})} + \frac{z'}{(\bar{z}"-\bar{z}')(\bar{z}-\bar{z}')} + \frac{z"}{(\bar{z}'-\bar{z}")(\bar{z}-\bar{z}")})\partial_z\partial_{z'}\partial_{z"} \tag{20}$$

etc....

Simple dimensional arguments at the level of the Hamiltonian are sufficient to understand the scaling property of the average density of states. Rescaling the length unit by λ amounts to rescale the Hamiltonian, thus the energy, by $1/\lambda^2$. On the other hand, in $d = 2$, the same length unit rescaling implies for the density ρ the same $1/\lambda^2$ rescaling. It is not surprising to find, after averaging, the E/ρ scaling of the density of states.

In the sequel one will concentrate only on the hard-core case, bearing in mind that the attractive-core case can be straightforwardly deduced from the hard-core case.

II.c Known results: the cases $\alpha \to 0$ and $\alpha = 1/2$

To simplify notations, we will use from now on the Brownian notations $e = m = 1$, and $\beta = t$.

From [2] we know that :

i) In the limit $\alpha \to 0$, one expects from (4) that $Z \to Z_{<B>} = Z_o < e^{i<B>\sum_n nS_n} >_{\{C\}}$, the partition function of one electron in an uniform magnetic field $< B >= \alpha\rho\phi_o$. However, possible corrections coming from the exponent $\exp(i2\pi n\alpha) - 1$ might alter this result. Due to the non-differentiability of Brownian paths, $\sum_n n^2 S_n$ is not defined for a typical Brownian curve where $< S_n >= t/(2\pi n^2)$ [5]. Certain recent results in the mathematical litterature [8] show that, for n sufficiently large, $n^2 S_n \to < n^2 S_n >= t/(2\pi)$. It follows that, when $\alpha \to 0$, $S \simeq < S > \simeq \pi\alpha$, and $A \simeq 2\pi\alpha\mathcal{A}$, where $\mathcal{A}$ is the algebraic area enclosed by the Brownian curve ($\mathcal{A} = \sum nS_n/t$). One deduces that $Z \to_{\alpha\to 0} Z_{<B>} e^{-t<B>/2}$, implying that the system of random vortices is equivalent to an uniform magnetic field $< B >$, but with an additional positive shift in the Landau spectrum $< B > /2 =< \omega_c >$. Note that $Z_{<B>}$ is built by the random variable A, and the shiff by the random variable S.

ii) When $\alpha = 1/2$ on the other hand, one can explicitly test the effect of the random distribution of vortices. (3) now reads $Z = Z_o < e^{-\rho tS} >_{\{C\}}$ and the average density of states, obtained by inverse Laplace transform of Z, is

$$< \rho(E) >= \rho_o(E) \int_0^{\frac{E}{\rho}} P(S) dS \tag{21}$$

$P(S)$, the probability distribution for the random variable S, was estimated numerically by simulations on a lattice, where a number of steps ranging from 2000 to 32000 was used. In Figure 1, $< \rho(E) >$ displays a Lifschitz tail at the bottom of the spectrum, around $E \simeq \rho < S >= \pi\rho/4$, where a behavior $< \rho(E) > \simeq \exp(-\rho/E)$ is expected.

In both these extreme cases $\alpha = 0, \alpha = 1/2$, an energy level depletion at the bottom of the spectrum is observed (global shift in the Landau spectrum in the former case, Lifschitz

tail in the latter). This pattern is quite reminiscent of one impurity $A-B$ density of states depletion $\rho_1(E) - \rho_o(E) = \frac{\alpha(\alpha-1)}{2}\delta(E)$ [9], which occurs precisely at $E = 0$.

The question we now ask is what happens when α continously decreases from $\alpha = 1/2$ to $\alpha = 0$? In particular, is it possible to understand the transition from a Lifschitz tail pattern to a Landau pattern in terms of several impurities average density of states?

III. Numerical evidences

III.a Specific heat

Interesting enough is the study of the specific heat averaged over disorder

$$c = kt^2 \frac{d^2}{dt^2} < \ln Z' >_{\{N_i\}} \tag{22}$$

where k is the Boltzman constant and

$$Z' = Z_o < e^{i\sum 2\pi n_i N_i \alpha} >_{\{C\}} \tag{23}$$

is the partition function for an electron in a given distribution of vortices $\{N_i\}$. We first note that

$$< \ln Z' >_{\{N_i\}} = \ln[< Z' >_{\{N_i\}}] (\equiv \ln Z) \tag{24}$$

In principle this property holds only for short ranged impurity potentials. This is of course not the case in the present problem. However, before averaging over disorder, the partition function involves only closed Brownian curves, and thus is entirely determined by the impurities distributed inside each Brownian curve (see (2)). Using the basic property that two Brownian curves on the plane have no chance to intersect each other, it is easy to get

$$< Z'^2 >_{\{N_i\}} = < Z' >^2_{\{N_i\}} \tag{25}$$

i.e. Z' is a self-averaging quantity. A straightforward computation leads to the t (inverse

temperature) expansion (note that $P(S,A) = P(S,-A)$)

$$c \simeq c_o + kt^2(< S^2 >_{\{C\}} - < S >^2_{\{C\}} - < A^2 >_{\{C\}}) + \cdots \tag{26}$$

Both quantities $< S^2 >_{\{C\}} - < S >^2_{\{C\}}$ and $< A^2 >_{\{C\}}$ have a natural interpretation in terms of a Landau spectrum : $< A^2 >_{\{C\}}$ determines the average magnetic field Landau levels, and $< S^2 >_{\{C\}} - < S >^2_{\{C\}}$ measures the deviation from the Landau pattern due to disorder. It is not a surprise that their difference $(< S^2 >_{\{C\}} - < S >^2_{\{C\}}) - < A^2 >_{\{C\}}$ plays a role in finding the critical point for the specific heat. They have been studied numerically, and are displayed in Fig. 2 for 2000 random walks of length $L = 100000$. When α continously decreases from $\alpha = 0.5$ to 0, $c - c_o$ is first positive, then vanishes for $\alpha \simeq 0.28$, then becomes negative. The vanishing of the t^2 term in the specific heat corresponds to a perfect gas behaviour. We will argue below, by analytical means, that the perfect gas behaviour takes place at $\alpha_o \simeq 0.29$.

III.b The density of states :

In order to estimate $< \rho(E) >$ numerically, a simple heuristic ansatz has to be made. Correlators between the variables S, A and the algebraic area $\mathcal{A} \equiv \sum_n nS_n/t$, such as

$$C_{S,A} \equiv \frac{< SA > - < S >< A >}{\sqrt{(< S^2 > - < S >^2)(< A^2 > - < A >^2)}} \tag{27}$$

or $C_{S,\mathcal{A}}$ are trivially vanishing because of obvious symmetry properties, and carry no information. It is more appropriate to use the variables $|A|$, and $|\mathcal{A}|$. Numerical simulations indicate that $C_{|A|,|\mathcal{A}|} \simeq 1$ for $\alpha < 0.05$, and $C_{|A|,|\mathcal{A}|} \simeq 0.93 - 0.95$ for $\alpha \simeq 0.25$. One deduces that the variables $|A|$ and $|\mathcal{A}|$ are highly correlated, in particular when α is very small, i.e. in the average magnetic field limit. We thus assume the linear relation

$$|A| = \frac{B}{\rho}|\mathcal{A}| \tag{28}$$

where B is an effective magnetic field determined by $B/\rho = \sqrt{\frac{<A^2>}{<\mathcal{A}^2>}} = \sqrt{12 < A^2 >}$, which gives, in the limit $\alpha \to 0$ that $B = 2\pi\alpha\rho =< B >$, as it should[4]. On the other hand, in the limit $\alpha \to 1/2$, $B = 0$, the magnetic field is absent, as again we already know from (4). In conclusion, the ansatz (28) seems quite reasonable. Now, introduce the positive random variable

$$S' = S - C_{S,|\mathcal{A}|}\sqrt{\frac{< S^2 > - < S >^2}{< \mathcal{A}^2 > - < |\mathcal{A}| >^2}}|\mathcal{A}| \tag{30}$$

so that S' and $|\mathcal{A}|$ are uncorrelated. The average partition function rewrites as

$$Z \simeq Z_o < e^{-\rho t S'} >< e^{\rho t(S'-S)} \cos(tB\mathcal{A}) > \tag{31}$$

The inverse Laplace transform of $Z/ < e^{-\rho t S'} >$ can be obtained analytically

$$\rho'(E') = \rho_o(E')\frac{1 - e^{-2u}}{1 + e^{-2u} + 2e^{-u}\cos\gamma} \tag{32}$$

where

$$\gamma = 2\pi\frac{(E'/\rho)(B/\rho)}{\mu^2 + (B/\rho)^2} \tag{33}$$

and

$$u = \frac{\gamma\mu}{(B/\rho)} \tag{34}$$

where $\mu = C_{S,|\mathcal{A}|}\sqrt{\frac{<S^2>-<S>^2}{<\mathcal{A}^2>-<|\mathcal{A}|>^2}}$. The average density of states rewrites

$$< \rho(E) >= \int_0^{E/\rho} P(S')\rho'(E - \rho S')dS' \tag{35}$$

[4]To obtain $< \mathcal{A}^2 >= 1/12$, consider simply the probability distribution for the random variable $\mathcal{A}$, which is nothing but the Fourier transform of the partition function of a charged particle in a constant magnetic field [5]

$$P(\mathcal{A}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dB e^{iB\mathcal{A}}\frac{B}{2\sinh B/2} \tag{29}$$

where $P(S')$ is the probability distribution for the variable S'. The numerical results are displayed in Fig. 3 for different values of α. It is clear that the average density of states is no more monotonic when α is below the critical value $\alpha_c \simeq 0.35$. Also, when $\alpha \to 0$, well separated Landau peaks appear with a well defined Landau gap. In this low energy region (which corresponds to long Brownian curves), the system clearly mimicks a constant average magnetic field.

IV. Analytical evidences

IV.a $\alpha_c > \alpha_o$:

Let us show that the transition for the density of states occurs necessarily at a critical $\alpha_c > \alpha_o$. Let us assume that when $E \to \infty$, $< \rho(E) > \to \rho_o(E) = \frac{V}{2\pi}$, i.e. at very high energy the system does not see the magnetic impurities, and when $E \to 0$, $< \rho(E) > \to 0$, i.e. at very small energy the depletion of states due to the hard-core impurities is effective. Both these assumptions are quite reasonnable, and are actually verified in all the numerical simulations and analytical studies. After integration by part in Z, one gets

$$c = c_o + \frac{kt^2}{2} \frac{\int_0^\infty \int_0^\infty dE dE' e^{-t(E+E')} \frac{d<\rho(E)/V>}{dE} \frac{d<\rho(E')/V>}{dE'} (E-E')^2}{\{\int_0^\infty e^{-tE} \frac{d<\rho(E)/V>}{dE}\}^2} \tag{36}$$

At small t, this expression becomes

$$c \simeq c_o + kt^2 2\pi^2 \int_0^\infty \int_0^\infty dE dE' \frac{d<\rho(E)/V>}{dE} \frac{d<\rho(E')/V>}{dE'} (E-E')^2 \tag{37}$$

which in turn implies that for $c - c_o$ to change its sign one necessarily should already have Landau oscillations.

IV.b The concentration expansion :

In the thermodynamic limit, the number of impurities is only fixed on average. We are in a situation similar to the one encountered in statistical mechanics for an undetermined number of particles, where there exists a cluster expansion for the thermodynamical potential, which involves at a given order of the fugacity z^N, the partition functions for N

particles, $N-1$ particles, etc... Here, there is one particle subject to the effect of an undetermined number of impurities. Again, there exists an expansion for the averaged one particle partition function [10], analog to the cluster expansion. The parameter of the expansion is the density (concentration) of impurities ρ, and at a given order ρ^N, the partition function per unit volume is expressed in terms of the partition functions for N impurities, $N-1$ impurities, etc... It reads

$$\begin{aligned} Z &= Z_0(1+\rho V(\frac{Z_1}{Z_0}-1)+\frac{1}{2!}\rho^2V^2[\frac{Z_2}{Z_0}-2\frac{Z_1}{Z_0}+1]+\frac{1}{3!}\rho^3V^3[\frac{Z_3}{Z_0}-3\frac{Z_2}{Z_0}+3\frac{Z_1}{Z_0}-1] \\ &+ \frac{1}{4!}\rho^4V^4[\frac{Z_4}{Z_o}-4\frac{Z_3}{Z_o}+6\frac{Z_2}{Z_o}-4\frac{Z_1}{Z_o}+1]+... \end{aligned} \tag{38}$$

where $Z_N \equiv < Tr\exp(-tH_N) > /V$.

Let us compare (38) with the ρt expansion of (4), namely

$$Z = Z_o(1-\rho t < S > +\frac{1}{2!}\rho^2t^2 < (S+iA)^2 > -\frac{1}{3!}\rho^3t^3 < (S+iA)^3 > +\cdots) \tag{39}$$

So, considering ρ as the expansion parameter, one sees that in order for both expansions (38) and (39) to match one necessarily has

$$\frac{Z_1}{Z_0}-1 = -\frac{t}{V} < S >, \quad \frac{Z_2}{Z_0}-1-2(\frac{Z_1}{Z_0}-1) = \frac{t^2}{V^2} < (S+iA)^2 >, \cdots \tag{40}$$

One already knows that

i) when $\alpha \to 0$, (39) is the partition function per unit volume of a charged particle in the average magnetic field, with a Landau spectrum shited by $< \omega_c >$

$$Z = \frac{1}{4\pi}\frac{< B >}{\sinh t < B > /2}e^{-t<B>/2} = \frac{1}{2\pi t}(1-\sum_{n=0}^{\infty}\frac{\zeta(-n)}{n!}(- < B > t)^{n+1}) \tag{41}$$

(again this follows directly from general properties of Brownian curves -note also that the attractive-core case would be obtained simply by $< B > \to - < B >$ in (41)).

ii) from the t expansion of the specific heat $c = c_o t^2 \frac{d^2 \ln Z}{dt^2}$ that the α_o value is attained when

$$< (S + iA)^2 >=< S >^2 \tag{42}$$

It follows that by a diagrammatic expansion in α of quantities such as $\frac{Z_1}{Z_0} - 1$, $\frac{Z_2}{Z_0} - 1 - 2(\frac{Z_1}{Z_0} - 1)$, $\cdots$, one should recover

i) the leading α behavior (41)

ii) the critical α_o value by finding the zeroes of $(\frac{Z_1}{Z_0} - 1)^2 = \frac{Z_2}{Z_0} - 1 - 2(\frac{Z_1}{Z_0} - 1)$

The diagrammatic expansion of Z_N is performed by usual perturbative methods for the Hamiltonian (11). One uses a non perturbed basis of free thermal propagators $G_t(\vec{r}, \vec{r}') = \frac{1}{2\pi t} \exp(-(\vec{r} - \vec{r}')^2/2t)$. In the computation of the average N impurity partition function, one encounters, at a given order in α, volume divergences from the $dz_i d\bar{z}_i dz_j d\bar{z}_j d...$ space integrals over the impurity locations $\vec{r}_i, \vec{r}_j, \cdots$, if computed directly in the thermodynamic limit. Thus, in principle, the need of a certain regularization prescription, as for example a harmonic regularization. However, in our case, one can circumvent this difficulty by computing, rather than average partition functions $Tr \exp(-tH_N)$, the average of the thermal propagators $G_N(\vec{r}_o, \vec{r}_o) \equiv < \vec{r}_o | \exp(-t\tilde{H}_N) | \vec{r}_o >$ from and to a given point $\vec{r}_o$. The actual average partition functions are by definition space integral over $\vec{r}_o$ of these propagators. Yet, averaging over disorder has to be made. If it is done before the final $\vec{r}_o$ space integration, it follows that $< G_N(\vec{r}_o, \vec{r}_o) >$ does not depend on $\vec{r}_o$ anymore. Therefore, the final space integration becomes trivial, since it amounts to multiply by V the coinciding point propagator. In this computational scheme, which relies on the crucial fact that averaging on the disorder has to be made, there is no need for a particular regularization procedure, since the infinite volume factors out trivially in the last space integral.

At a given order α^n, a diagram has n vertex, and $m \leq n$ impurities: it contributes at

order $\alpha^n \rho^m$ in the concentration expansion. The diagrams which contribute to the $< B >$ average partition function are necessarily such that $m = n$, and corrections to the average field limit are built by diagrams with $m < n$. We have analytically computed the leading $< B >$ diagrams up to order $\alpha^4 \rho^4$, and at order ρ^2 -the two impurity case-, the corrections to the leading $\alpha^2 \rho^2$ digram up to order α^4. The diagrammatic expansion results are:

$$\frac{Z_1}{Z_0} - 1 = \frac{1}{VZ_o} \frac{\alpha(\alpha - 1)}{2} \tag{43}$$

$$\frac{Z_2}{Z_0} - 1 - 2(\frac{Z_1}{Z_0} - 1) = \frac{1}{(VZ_o)^2} (\frac{1}{6}\alpha^2 + 0\alpha^3 + (\frac{1}{24} - \frac{7}{16}\zeta(3))\alpha^4 + \cdots) \tag{44}$$

where $\frac{1}{24} - \frac{7}{16}\zeta(3) = -0.48423323$,

$$\frac{Z_3}{Z_0} - 3\frac{Z_2}{Z_0} + 3\frac{Z_1}{Z_0} - 1 = \frac{1}{(VZ_o)^3} (0\alpha^3 + \cdots) \tag{45}$$

$$\frac{Z_4}{Z_0} - 4\frac{Z_3}{Z_0} + 6\frac{Z_2}{Z_0} - 4\frac{Z_1}{Z_0} + 1 = \frac{1}{(VZ_o)^4} (- \frac{1}{30}\alpha^4 + \cdots) \tag{46}$$

We see that at leading order $\alpha^n \rho^n$, (43, 44, 45, 46) indeed reproduce the average magnetic field expansion (41).

We are now in position to determine the value α_o at which the specific heat transition occurs, just by considering (43,44) on the one hand, (40,42) on the other hand. Before doing so, the diagrammatic expansion (44) should be first completed to obtain a $\alpha(1-\alpha)$ polynomial, since this should be so. One gets

$$\frac{Z_2}{Z_0} - 1 - 2(\frac{Z_1}{Z_0} - 1) = \frac{1}{(VZ_o)^2} \left(\frac{1}{6}[\alpha(1-\alpha)]^2 + \frac{1}{3}[\alpha(1-\alpha)]^3 + \frac{7}{8}(1 - \frac{1}{2}\zeta(3))[\alpha(1-\alpha)]^4 + O([\alpha(1-\alpha)]^5) \right) \tag{47}$$

The resulting polynomial equation in α writes

$$(\frac{\alpha(1-\alpha)}{2})^2 = \frac{1}{6}[\alpha(1-\alpha)]^2 + \frac{1}{3}[\alpha(1-\alpha)]^3 + \frac{7}{8}(1 - \frac{1}{2}\zeta(3))[\alpha(1-\alpha)]^4 \tag{48}$$

In the interval $\alpha \in [0, 1/2]$, this equation has the desired root $\alpha_o = 0.29$.

One can go a bit further, and compare the diagrammatic expansion with the numerical estimations for $(< S^2 >_{\{C\}} - < S >^2_{\{C\}}) - < A^2 >_{\{C\}})$. In Fig. 4, not only the three curves - a) and b) are numerical simulations, c) is the diagrammatic expansion- have the same intercept with the horizontal axis (thus the same $\alpha_o \simeq \alpha_o^{num}$ value), but also they exhibit the same qualitative behavior. Differences are due to the fact that actual winding properties for true Brownian walks are difficult to reach from finite length random walks, and to the fact that the perturbative expansion in $\alpha(1-\alpha)$ is by definition incomplete. Still, it is remarkable that both approaches yield the same value for α_o, altogether with the same α qualitative behavior. A convergence between both approaches is expected.

An interesting consequence of the discussion above concerns the average density of states for the one impurity problem, two impurities problem, etc. In the simple one impurity case, averaging over disorder is trivial, since the Aharonov-Bohm low energy depletion of states does not depend on the position of the vortex. One simply recovers the standard result [9]

$$< \rho_1(E) - \rho_o(E) >= \rho_1(E) - \rho_o(E) = \frac{\alpha(\alpha-1)}{2}\delta(E) \tag{49}$$

In the two-impurities case, things become highly non trivial, but averaging has allowed for the diagrammatic expansion result (44). One deduces that

$$\begin{aligned} < \rho_2(E) - \rho_o(E) > - \quad 2 \quad < \rho_1(E) - \rho_o(E) >= \frac{2\pi}{V}\Big(\frac{1}{6}[\alpha(1-\alpha)]^2 + \frac{1}{3}[\alpha(1-\alpha)]^3 \\ + \quad \frac{7}{8}(1-\frac{1}{2}\zeta(3))[\alpha(1-\alpha)]^4 + \dots\Big)\delta'(E) \end{aligned} \tag{50}$$

$$< \rho_3(E) - \rho_o(E) > -3 < \rho_2(E) - \rho_o(E) > +3 < \rho_1(E) - \rho_o(E) >= (\frac{2\pi}{V})^2(0\alpha^3 + \cdots)\delta''(E) \tag{51}$$

$$< \rho_4(E) - \rho_o(E) > -4 < \rho_3(E) - \rho_o(E) > \quad + \quad 6 < \rho_2(E) - \rho_o(E) > -4 < \rho_1(E) - \rho_o(E) >=$$
$$(\frac{2\pi}{V})^3(-\frac{1}{30}\alpha^4 \quad + \quad \cdots)\delta'''(E) \tag{52}$$

etc... At leading order $\rho^n\alpha^n$, summing up all leading contributions from the one, two, $\cdots$ impurity cases, leads to

$$\begin{aligned} < \rho(E) - \rho_o(E) > &= \frac{V < B >}{2\pi}\Big(-\frac{1}{2}\delta(E) + \frac{< B >}{2!}\frac{1}{6}\delta'(E) \\ &\quad + \frac{< B >^2}{3!}0\delta''(E) + \frac{< B >^3}{4!}(-\frac{1}{30})\delta'''(E) + \cdots \Big) \\ &= \frac{V < B >}{2\pi}\sum_{n=1}^{\infty}\delta(E - n < B >) - \rho_o(E) = \rho_L(E) - \rho_o(E) \end{aligned} \tag{53}$$

which reproduces, as expected, the shifted Landau density of states $\rho_L(E) - \rho_o(E)$.

IV.c Starting from the Landau basis:

Perturbative computations can also be done by starting directly from the Hamiltonian (11). The expansion of the average partition function will be modified in the following way:

- propagators to be used are Landau propagators in the average magnetic field

$$G_t(r, r') = \frac{< \omega_c >}{2\pi \sinh t < \omega_c >} e^{-\frac{< \omega_c >}{2 \sinh t < \omega_c >}[|z - z'|^2 \cosh t < \omega_c > + \sinh t < \omega_c > (z\bar{z}' - \bar{z}z')]}$$

instead of free propagators.

- the vertex to be used is

$$-2\alpha\frac{1}{\bar{z} - \bar{z}'_i}(\partial_z - \frac{1}{2} < \omega_c > \bar{z}) + 2\alpha\frac{\pi\rho}{N}(z\partial_z - \frac{1}{2} < \omega_c > z\bar{z})$$

instead of

$$-2\alpha\frac{\partial_z}{\bar{z} - \bar{z}_i}$$

One has already noted that any diagram with isolated impurities only, i.e. of the $\alpha^n\rho^n$ type, contributes to the average magnetic field partition function. It follows that, by definition, it vanishes in the present formulation since the average magnetic field contributions are ab initio incorporated in the Landau propagator. It is possible to show that for any subleading diagrams $\alpha^n\rho^m, m < n$, the vertex reduces to

$$-2\alpha\frac{1}{\bar{z}-\bar{z}_i}(\partial_z - \frac{1}{2} < \omega_c > \bar{z})$$

Any such subleading diagram automatically incorporates the corresponding diagram in the free propagator approach, plus all diagrams deduced from it by adding an arbitrary number of isolated impurity lines, i.e. diagrams of the type $\alpha^n\rho^m(1+\alpha\rho+\alpha^2\rho^2+\cdots), m < n$.

V. Conclusion

A more precise value of α_c and a better understanding of what is exactly happening at the transition is still missing. How Landau levels actually appear when α continuously decreases from α_c to 0? What is the exact nature of the transition?

Another important issue concerns the conductivity and localisation properties of the test particle in the random system of magnetic impurities. In particular, one would like to have information on the Hall conductance in the impurity system. This question could be partially adressed by considering the average persistent currents due to the vortex distribution[11]. One has been able to get perturbative expansion in α for the one, two, $\cdots$ impurity average densities of states. In principle, one can deduce [11] from these density of states the average persistent currents for one vortex, two vortices, $\cdots$. In the limit $\alpha \to 0$, at leading order $\alpha^n\rho^n$, diamagnetic persistent currents for a constant magnetic field should be recovered by summing up all contributions coming from the N impurities persistent currents.

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Figure captions :

Figure 1: The average level density of states $< \rho(E) >$ as a function of the variable E/ρ for $\alpha = 1/2$ exhibits a Lifshitz tail at the bottom of the spectrum.

Figure 2: The variances of the random variables S and A obtained by simulations (2000 closed random walks of 100000 steps) are plotted as a function of α. The intersection of the two curves at $\alpha_o^{num} = 0.28$ determines a change of the specific heat behavior at high temperature.

Figure 3: The average level density of states, determined by numerical simulations, as a function of the variable E/ρ for different α values. When $\alpha < \alpha_c \simeq 0.35$, the density of states is no more monotonic and oscillations appear.

Figure 4: The difference between the variances of S and A as a function of α, determined by numerical simulations for a) 3000 closed random walks of 400 steps, b) 2000 closed random walks pf 100000 steps, and also by the diagrammatic expansion to fourth order in α. The three curves intercept the horizontal axis at $\alpha \simeq 0.28 - 0.30$. The origin of the quantitative differences are explained in the text.